PERGAMON

International Journal of Solids and Structures 40 (2003) 1455–1463

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Hencky's logarithmic strain and dual stress–strain and strain–stress relations in isotropic finite hyperelasticity

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Received 5 June 2002; received in revised form 18 October 2002

Dedicated to Professor Wei-zang Chien on the occasion of his 90th birthday

Abstract

It has been known that the Kirchhoff stress tensor $\boldsymbol{\tau}$ and Hencky's logarithmic strain tensor $\boldsymbol{h}$ may be useful in formulations of isotropic finite elasticity and elastoplasticity. In this work, a straightforward proof is presented to demonstrate that, for an isotropic hyperelastic solid, the just-mentioned stress–strain pair $\boldsymbol{\tau}$ and $\boldsymbol{h}$ are derivable from two dual scalar potentials with respect to each other. These results establish a simple, explicit dual formulation of isotropic finite hyperelasticity. As a result, they supply a complete solution to the problem of finding out the inverted stress–strain relation for isotropic hyperelastic solids, raised by J.A. Blume [Int. J. Non-linear Mech. 27 (1992) 413]. Moreover, an explicit form of such an inverted hyperelastic stress–strain relation is derived in terms of the powers $\boldsymbol{I}$, $\boldsymbol{\tau}$ and $\boldsymbol{\tau}^2$.

Keywords: Finite strain; Hyperelasticity; Stress–strain relation; Strain–stress relation; Duality

1. Introduction

Usually, an isotropic elastic solid undergoing finite deformations is defined by a stress–strain relation that prescribes the dependence of a stress measure $\boldsymbol{T}$ on a strain measure $\boldsymbol{E}$, i.e.,

$$\boldsymbol{T} = \widehat{T}(\boldsymbol{E}).$$

For a hyperelastic solid, certain restrictions concerning the specific stress power should be imposed. As a result, the stress–strain relation for isotropic finite hyperelastic solids can be derived from an isotropic scalar potential known as the strain-energy function.

On the other hand, an isotropic elastic solid may be equally well defined by an inverted stress–strain relation, i.e., a strain–stress relation, which gives the dependence of a strain measure $\boldsymbol{E}$ on a stress measure $\boldsymbol{T}$, i.e.,

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0020-7683/02/$ - see front matter © 2002 Elsevier Science Ltd. All rights reserved.
doi:10.1016/S0020-7683(02)00653-4

$$E = \widehat{E}(T).$$

Because of the foregoing restrictions, the strain–stress relation for isotropic hyperelastic solids should also be derived from a scalar potential.

In continuum mechanics, there are a great many stress and strain measures for consideration. In formulating an elastic relation, the stress measure $\boldsymbol{T}$ and the strain measure $\boldsymbol{E}$ may be freely chosen among them in principle. The Cauchy stress tensor $\boldsymbol{\sigma}$ and the Cauchy–Green tensor $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}$ and the stretch tensor $\boldsymbol{V} = \sqrt{\boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}}$ are commonly used in literature. With the pair $(\boldsymbol{\sigma}, \boldsymbol{B})$ or $(\boldsymbol{\sigma}, \boldsymbol{V})$, the explicit stress–strain relation $\boldsymbol{\sigma} = \hat{\sigma}(\boldsymbol{B})$ for isotropic hyperelastic solids in terms of the strain-energy function is well-known (see, e.g., Truesdell and Noll, 1965; Gurtin, 1981; Ogden, 1984). It may be clear that the form of a hyperelastic stress response function $\widehat{T}(\boldsymbol{E})$ or a hyperelastic strain response function $\widehat{E}(\boldsymbol{T})$ relies on the choice of the stress–strain pair $(\boldsymbol{T}, \boldsymbol{E})$. Generally, for a stress–strain pair $(\boldsymbol{T}, \boldsymbol{E})$ that need not be work-conjugate, the explicit form of the hyperelastic stress–strain relation $\boldsymbol{T} = \widehat{T}(\boldsymbol{E})$ may not be so clear or so simple. In particular, for a chosen stress measure $\boldsymbol{T}$, a given form of stress response function $\widehat{T}(\boldsymbol{E})$ may not be well-defined in the sense of hyperelasticity for every strain measure $\boldsymbol{E}$. Most recently, Chiskis and Parnes (2000) have studied an interesting particular example in this respect. Let the stress measure $\boldsymbol{T}$ be the Cauchy stress $\boldsymbol{\sigma}$, i.e., $\boldsymbol{T} = \boldsymbol{\sigma}$, and let the strain measure $\boldsymbol{E}$ to be determined. They consider a Hookean type elastic relation linear in $\boldsymbol{E}$, i.e.,

$$\boldsymbol{\sigma} = \Lambda(\mathrm{tr}\,\boldsymbol{E})\boldsymbol{I} + 2G\boldsymbol{E}, \tag{1}$$

where Λ and G are the Lamé elastic constants evaluated at small deformations. They demonstrate that the elastic relation (1) is hyperelastic if and only if $\boldsymbol{E}$ is of the form

$$\boldsymbol{E} = \frac{\Lambda}{2G}\left(\boldsymbol{V}^{(2G/\Lambda)} - \boldsymbol{I}\right).$$

The latter requires that $\boldsymbol{E}$ be dependent on the Lamé elastic constants Λ and G. It does not appear that such an $\boldsymbol{E}$ qualifies as a strain measure in a pure kinematic sense, since it specifies different straining states for the same deformation of material bodies with different Lamé constants.

In a paper, Blume (1992) raised and investigated the problem of finding out an explicit representation for the strain–stress relation [1] $\boldsymbol{B} = \widehat{B}(\boldsymbol{\sigma})$ for isotropic hyperelastic solids. She derived conditions on the form of such an inverted hyperelastic constitutive relation, and, in the incompressible case, achieved an explicit general representation for a hyperelastic strain–stress relation in terms of a generating scalar potential. However, it appears that no explicit results have been derived for the general compressible case. Moreover, Blume (1992) noted that, even for a simple incompressible case, it appears to be difficult to derive an explicit form of the hyperelastic strain–stress relation in terms of the powers $\boldsymbol{\sigma}^r$ with $r = 0, 1, 2$.

The known stress measure closest to the Cauchy stress (true stress) $\boldsymbol{\sigma}$ is the Kirchhoff stress $\boldsymbol{\tau} = J\boldsymbol{\sigma}$, also known as the weighted Cauchy stress. We shall show that, if we replace the Cauchy stress $\boldsymbol{\sigma}$ with the Kirchhoff stress $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ in the aforementioned issues raised by Chiskis and Parnes (2000) and Blume (1992), respectively, then simple, complete solutions for them would be possible. With the replacement of the Cauchy stress $\boldsymbol{\sigma}$ by the Kirchhoff stress $\boldsymbol{\tau} = J\boldsymbol{\sigma}$, Eq. (1) becomes

$$\boldsymbol{\tau} = J\boldsymbol{\sigma} = \Lambda(\mathrm{tr}\,\boldsymbol{E})\boldsymbol{I} + 2G\boldsymbol{E}. \tag{2}$$

Then arises the question as to what strain measure makes the above Hookean type elastic relation hyperelastic. Moreover, replacing the Cauchy stress $\boldsymbol{\sigma}$ by the Kirchhoff stress $\boldsymbol{\tau} = J\boldsymbol{\sigma}$, we may reformulate the foregoing Blume's problem as follows. Let $\boldsymbol{B} = \overline{B}(\boldsymbol{\tau})$ be an isotropic hyperelastic strain–stress relation. Find

[1] Note that the Cauchy–Green strain tensor $\boldsymbol{G}$ used in Blume (1992) is replaced here by $\boldsymbol{B}$ and that the symbol $\boldsymbol{\tau}$ for the Cauchy stress therein is changed to $\boldsymbol{\sigma}$ here. In this article the symbol $\boldsymbol{\tau}$ is used to designate the Kirchhoff stress, as will be indicated.

a general explicit expression for the strain response function $\overline{B}(\tau)$ in terms of a free scalar potential $\widetilde{\Sigma} = \overline{\Sigma}(\tau)$. Note that the Kirchhoff stress τ is just the Cauchy stress scaled by the Jacobian (volume ratio) $J = \det \boldsymbol{F}$. They differ only by a scalar factor J. Since the stress power is just the inner product of the Kirchhoff stress τ and the stretching $\boldsymbol{D}$ and since the notion of hyperelasticity is concerned directly with the stress power, it may be expected that the Kirchhoff stress τ should be more pertinent than the Cauchy stress σ in finite hyperelastic formulation.

In this work, we shall show that, with the Kirchhoff stress tensor τ and Hencky's logarithmic strain measure $\boldsymbol{h}$ (see Eqs. (3) and (4) below) we may arrive at a simple, explicit dual formulation of stress–strain and strain–stress relations for isotropic finite hyperelasticity. We demonstrate in a straightforward manner that, for an isotropic hyperelastic solid, the foregoing stress–strain pair τ and $\boldsymbol{h}$ are derivable from two dual scalar potentials with respect to each other. These results supply a complete solution to the foregoing problem raised by Blume (1992). In particular, with reference to the foregoing issue treated by Chiskis and Parnes (2000), we show that the linear stress–strain relation between the Kirchhoff stress and Hencky strain is hyperelastic for any given Lamé constants. Moreover, using the eigenprojection method based on Sylvester's formula, we derive an explicit form of the hyperelastic strain–stress relation in terms of the three powers τ^r with $r = 0, 1, 2$.

Usefulness of the inverted stress–strain relation for hyperelastic solids has been pointed out by Blume (1992); refer to the relevant references therein. In addition, in recent years, a hyperelastic strain–stress relation in terms of the Hencky strain and Kirchhoff stress has been found essential to formulations of finite inelasticity theories (see, e.g., Bruhns et al., 1999, 2001b; Xiao et al., 1997a,b, 1999, 2000). Generally, Hill (1968, 1970, 1978) found that Hencky's logarithmic strain measure has inherent advantages over other strain measures in his study of a priori constitutive inequalities [2] and treated the Hencky strain, its rate and its work-conjugate stress as basic measures for strain, strain rates and stresses, etc. Recently, certain significant properties of the Hencky strain or natural strain have been indicated by Freed (1995), Bažant (1998), and Xiao et al. (1997b). Now, the Hencky strain has found applications in finite elasticity and inelasticity; refer to, e.g., the relevant references mentioned above, as well as de Boer (1967), de Boer and Bruhns (1969), Bruhns and Thermann (1969), Bruhns (1970, 1971), Stören and Rice (1975), Raniecki and Nguyen (1984), Eterovic and Bathe (1990), Weber and Anand (1990), Miehe et al. (1994), Stumpf and Schieck (1994), Schieck and Stumpf (1995), Bonet and Wood (1997), Kollmann and Sansour (1997), Miehe (1998), and many others.

2. Dual stress–strain and strain–stress relations

Let τ be the Kirchhoff stress tensor and $\boldsymbol{h}$ Hencky's logarithmic strain tensor. [3] The former is simply the Cauchy stress multiplied by the volume ratio J, and the latter is the natural logarithm of the left stretch tensor $\boldsymbol{V}$. Namely, [4]

$$\tau = J\sigma, \quad J = \det \boldsymbol{F}, \tag{3}$$

[2] In this respect, refer to Ogden (1970) for a further study.

[3] According to literature, the logarithmic strain measure, also called natural strain, is named after Hencky. However, it was introduced earlier by several researchers, including Imbert (1880) and Ludwik et al. (1909) (see, e.g., Truesdell and Noll, 1965; Curnier and Rakotomanana, 1991). Later Hencky (1928, 1931, 1933) independently introduced it and used it to study elastic behaviour of rubbers etc. at some simple finite deformation modes.

[4] A general class of finite strain measures including the Hencky strain $\boldsymbol{h}$ was introduced by Hill (1968, 1970, 1978). A coherent, comprehensive treatment for them can be found in Ogden (1984).

$$\boldsymbol{h} = \ln \boldsymbol{V} = \frac{1}{2} \ln \boldsymbol{B} = \sum_{i=1}^{3} (\ln \lambda_i) \boldsymbol{n}_i \otimes \boldsymbol{n}_i. \tag{4}$$

Here and henceforward, $\boldsymbol{F}$ is the deformation gradient, $\boldsymbol{V} = \sqrt{\boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}}$ as before, and λ_i and $\boldsymbol{n}_i$ are the three eigenvalues of $\boldsymbol{V}$ and the corresponding subordinate orthonormal eigenvectors of $\boldsymbol{V}$. The latter obey the orthonormalization condition

$$\boldsymbol{n}_i \cdot \boldsymbol{n}_j = \delta_{ij}, \quad i, j = 1, 2, 3. \tag{5}$$

The constitutive equations for isotropic elastic materials can be defined by a tensor function relation between a stress measure $\boldsymbol{T}$ and a strain measure $\boldsymbol{E}$, as indicated before. In particular, we have

$$\boldsymbol{\tau} = \boldsymbol{\Phi}(\boldsymbol{h}) \tag{6}$$

satisfying the following isotropy condition:

$$\boldsymbol{\Phi}(\boldsymbol{Q}\boldsymbol{h}\boldsymbol{Q}^{\mathrm{T}}) = \boldsymbol{Q}\boldsymbol{\Phi}(\boldsymbol{h})\boldsymbol{Q}^{\mathrm{T}} \tag{7}$$

for every orthogonal tensor $\boldsymbol{Q}$. According to Rivlin–Ericksen representation theorem derived from Cayley–Hamilton theorem (see, e.g., Truesdell and Noll, 1965), we have

$$\boldsymbol{\tau} = \boldsymbol{\Phi}(\boldsymbol{h}) = a\boldsymbol{I} + b\boldsymbol{h} + c\boldsymbol{h}^2, \tag{8}$$

where a, b and c are symmetric functions of the three eigenvalues $h_i = \ln \lambda_i$ of $\boldsymbol{h}$.

For the stretching $\boldsymbol{D}$ holds the following important fact (see, e.g., Hill, 1978; Ogden, 1984),

$$\boldsymbol{n}_i \cdot \boldsymbol{D}\boldsymbol{n}_i = \frac{\dot{\lambda}_i}{\lambda_i} \quad \text{(no summation)}. \tag{9}$$

Here and henceforth, the symbol $\dot{(\ )}$ with a superposed dot is used to represent the material time derivative.

For any given non-negative integer r, we have

$$\boldsymbol{h}^r = \sum_{i=1}^{3} (\ln^r \lambda) \boldsymbol{n}_i \otimes \boldsymbol{n}_i. \tag{10}$$

The material time derivative of the Hencky strain tensor $\boldsymbol{h}$ given by Eq. (4) is of the form:

$$\dot{\boldsymbol{h}} = \sum_{i=1}^{3} \left(\frac{\dot{\lambda}_i}{\lambda_i} \boldsymbol{n}_i \otimes \boldsymbol{n}_i + \ln \lambda_i (\boldsymbol{n}_i \otimes \dot{\boldsymbol{n}}_i + \dot{\boldsymbol{n}}_i \otimes \boldsymbol{n}_i) \right). \tag{11}$$

Since $\boldsymbol{n}_i \cdot \boldsymbol{n}_i = 1$ (no summation), we have

$$\boldsymbol{n}_i \cdot \dot{\boldsymbol{n}}_i = 0 \quad \text{(no summation)}. \tag{12}$$

Hence, with Eqs. (5) and (10)–(12) we deduce [5]

$$\mathrm{tr}(\boldsymbol{h}^r \boldsymbol{D}) = \mathrm{tr}(\boldsymbol{h}^r \dot{\boldsymbol{h}}) \tag{13}$$

for any given non-negative integer r. Then, the identity (13) and the representation formula (8) together produce the stress power $\dot{w}$ per unit reference volume as follows:

$$\dot{w} = \mathrm{tr}(\boldsymbol{\tau}\boldsymbol{D}) = \mathrm{tr}(\boldsymbol{\tau}\ \dot{\boldsymbol{h}}) \tag{14}$$

for an isotropic elastic material.

[5] Throughout, the notation $\mathrm{tr}\boldsymbol{A}$ with a 2nd-order tensor $\boldsymbol{A}$ is used to represent the trace of $\boldsymbol{A}$, i.e., $\mathrm{tr}\boldsymbol{A} = A_{ii}$.

According to the definition of hyperelasticity or Green elasticity, there is an isotropic scalar potential [6]

$$\Sigma = \widehat{\Sigma}(\boldsymbol{h}) \tag{15}$$

with

$$\widehat{\Sigma}(\boldsymbol{Q}\boldsymbol{h}\boldsymbol{Q}^{\mathrm{T}}) = \widehat{\Sigma}(\boldsymbol{h})$$

for every proper orthogonal tensor $\boldsymbol{Q}$, known as strain energy function, such that the material time derivative $\dot{\Sigma}$ supplies the stress power $\dot{w}$ per unit reference volume, i.e.,

$$\dot{\Sigma} = \operatorname{tr}(\boldsymbol{\tau}\boldsymbol{D}). \tag{16}$$

From Eqs. (14)–(16) it may become clear that an isotropic elastic material is hyperelastic if and only if the Kirchhoff stress $\boldsymbol{\tau}$ is derivable from the isotropic scalar potential $\Sigma = \widehat{\Sigma}(\boldsymbol{h})$ with respect to the Hencky strain $\boldsymbol{h}$, i.e.,

$$\boldsymbol{\tau} = \frac{\partial \Sigma}{\partial \boldsymbol{h}}. \tag{17}$$

On the other hand, if either we assume that the scalar potential $\Sigma = \widehat{\Sigma}(\boldsymbol{h})$ is twice continuously differentiable and that the Hessian $(\partial^2\Sigma/\partial\boldsymbol{h}\partial\boldsymbol{h})$ is non-singular, or we assume that $\Sigma = \widehat{\Sigma}(\boldsymbol{h})$ is continuously differentiable and very strictly convex (see, e.g., Definition 16.2.7, Šilhavý, 1997), then (17) is invertable to yield the inverted relation

$$\boldsymbol{h} = \hat{\boldsymbol{h}}(\boldsymbol{\tau}) \tag{18}$$

and, therefore, we may define the Legendre transformation $\widetilde{\Sigma}$ of Σ as follows:

$$\widetilde{\Sigma} = \overline{\Sigma}(\boldsymbol{\tau}) = \operatorname{tr}(\boldsymbol{\tau}\hat{\boldsymbol{h}}(\boldsymbol{\tau})) - \widehat{\Sigma}(\hat{\boldsymbol{h}}(\boldsymbol{\tau})). \tag{19}$$

We refer to $\widetilde{\Sigma}$ as the complementary potential. It is also twice continuously differentiable or continuously differentiable (see, e.g., Šilhavý, 1997, Proposition 10.1.3 and Corollary 16.4.7).

The potential Σ and $\overline{\Sigma}$ form a dual or conjugate relation. Indeed, we have

$$\overline{\Sigma}(\boldsymbol{\tau}) + \widehat{\Sigma}(\boldsymbol{h}) = \operatorname{tr}(\boldsymbol{\tau}\boldsymbol{h}) \tag{20}$$

and, besides, we have (17) and

$$\boldsymbol{h} = \frac{\partial \widetilde{\Sigma}}{\partial \boldsymbol{\tau}}. \tag{21}$$

The latter indicates that, for an isotropic hyperelastic material, the Hencky strain $\boldsymbol{h}$ is derivable from the complementary potential $\widetilde{\Sigma}$ with respect to the Kirchhoff stress $\boldsymbol{\tau}$.

By the above analysis we have established explicit dual formulations of stress–strain and strain–stress relations for isotropic hyperelastic materials in terms of the potential $\Sigma = \widehat{\Sigma}(\boldsymbol{h})$ and the complementary potential $\widetilde{\Sigma} = \overline{\Sigma}(\boldsymbol{\tau})$ which are related to each other through the Legendre relation (20). The hyperelastic stress–strain relation (17) was known to Hill (1968, 1970, 1978). A detailed proof was given latter by Fitzjerald (1980) and Hoger (1987). Most recently, a novel, concise proof has been presented by Sansour (2001). The dual formulation formed by Eqs. (17) and (21) was established by Bruhns et al. (1999) and Xiao et al. (2000) by applying the integrability theorem derived in Xiao et al. (1997a). As has been shown above, here these results may be established by means of a more straightforward procedure.

[6] Note that this potential may be formulated as an isotropic scalar function of any chosen strain measure.

The strain–stress relation (21) is expressible as another useful form. In fact, substituting Eq. (21) into the identity

$$\boldsymbol{B} = \boldsymbol{V}^2 = \mathrm{e}^{2\boldsymbol{h}}, \tag{22}$$

we obtain

$$\boldsymbol{B} = \mathrm{e}^{2(\partial\tilde{\Sigma}/\partial\boldsymbol{\tau})}. \tag{23}$$

In the above, $\mathrm{e}^{\boldsymbol{A}}$ is used to designate the exponential function of symmetric second order tensor $\boldsymbol{A}$.

For an incompressible material, we have $J = 1$. In this case, the Kirchhoff stress $\boldsymbol{\tau}$ coincides with the Cauchy stress $\boldsymbol{\sigma}$. Thus, Eq. (23) provides the solution derived by Blume (1992) for incompressible materials. It may be interesting to observe that Blume's solution (cf. Eq. (3.10) in Blume, 1992) for the incompressible case yields the general solution (23) simply by replacing the potential Γ and the Cauchy stress therein with the complementary potential $\tilde{\Sigma}$ and the Kirchhoff stress $\boldsymbol{\tau}$ here.

On the other hand, we consider the issue concerning what strain measure $\boldsymbol{E}$ makes the Hookean type relation (2) hyperelastic. With the quadratic strain-energy function

$$\Sigma = \tfrac{1}{2}\Lambda(\mathrm{tr}\,\boldsymbol{h})^2 + G(\mathrm{tr}\,\boldsymbol{h}^2), \tag{24}$$

the general hyperelastic stress–strain relation (17) yields the following linear relation between the Kirchhoff stress $\boldsymbol{\tau}$ and Hencky strain measure $\boldsymbol{h}$:

$$\boldsymbol{\tau} = J\boldsymbol{\sigma} = \Lambda(\mathrm{tr}\,\boldsymbol{h})\boldsymbol{I} + 2G\boldsymbol{h}. \tag{25}$$

As the simplest case of the general hyperelastic relation (17), the above linear stress–strain relation is hyperelastic for any given Lamé elastic constants Λ and G. Unlike the solution given by Chiskis and Parnes (2000), here there appears no strain measure that depends on the elastic constants.

The linear hyperelastic stress–strain relation (25) was introduced and used by Hencky (1928, 1931, 1933) about 75 years ago. Later, de Boer (1967), de Boer and Bruhns (1969), Bruhns and Thermann (1969), and Bruhns (1970, 1971) used Eq. (25) to study finite bending deformations of incompressible and compressible elastic and elastoplastic plate strips. In recent years, certain remarkable properties of this simple relation have been uncovered both from experimental grounds by Anand (1979, 1986) and Bruhns et al. (2001a) and from theoretical grounds relating to the exact integrability of the widely used zeroth-grade hypoelastic equation with objective stress rates (see, e.g., Xiao et al., 1999, 2000).

In a general respect, it has been shown by Bruhns et al. (1999) and Xiao et al. (2000) that the general hyperelastic strain–stress relation (21) leads to a simple form of explicit, integrable-exactly rate formulation of hypoelastic type of the general isotropic finite hyperelasticity, in which the instantaneous tangential elastic compliance tensor is exactly the twice derivative of the complementary potential $\tilde{\Sigma}$, i.e., the Hessian $(\partial^2\tilde{\Sigma}/\partial\boldsymbol{\tau}\partial\boldsymbol{\tau})$. Namely, we have

$$\boldsymbol{D} = \frac{\partial^2\tilde{\Sigma}}{\partial\boldsymbol{\tau}\partial\boldsymbol{\tau}} : \overset{\circ}{\boldsymbol{\tau}}{}^{\log}. \tag{26}$$

In the above, $\overset{\circ}{\boldsymbol{\tau}}{}^{\log}$ is the logarithmic rate of the Kirchhoff stress tensor $\boldsymbol{\tau}$, which is an objective co-rotational rate and introduced in Xiao et al. (1997b). Eq. (26) is just an equivalent Eulerian rate form of the general hyperelastic strain–stress relation (21). The uniqueness property of this rate form and further properties have been discussed in Bruhns et al. (1999) and Xiao et al. (2000).

3. Explicit strain–stress relation in terms of the powers $\boldsymbol{\tau}^r$

Since $\overline{\Sigma}(\boldsymbol{\tau})$ is isotropic, it is a symmetric function of the three eigenvalues τ_i of $\boldsymbol{\tau}$, i.e.,

$$\overline{\Sigma}(\boldsymbol{\tau}) = \psi(\tau_1, \tau_2, \tau_3).$$

Hence we have [7]

$$\frac{\partial\widetilde{\Sigma}}{\partial\boldsymbol{\tau}} = \sum_{i=1}^{3}\frac{\partial\psi}{\partial\tau_i}\boldsymbol{n}_i\otimes\boldsymbol{n}_i. \tag{27}$$

Thus, Eq. (23) has the spectral form

$$\boldsymbol{B} = \sum_{i=1}^{3} e^{2(\partial\psi/\partial\tau_i)}\boldsymbol{n}_i\otimes\boldsymbol{n}_i. \tag{28}$$

The latter yields

$$\lambda_i = e^{(\partial\psi/\partial\tau_i)}. \tag{29}$$

Sometimes, we need to put Eq. (23) in the usual power form

$$\boldsymbol{B} = b_0\boldsymbol{I} + b_1\boldsymbol{\tau} + b_2\boldsymbol{\tau}^2, \tag{30}$$

where the coefficients b_s with $s = 0, 1, 2$ are three symmetric functions of the three eigenvalues τ_i of $\boldsymbol{\tau}$. Usually, it does not appear to be easy to work out such a form, as remarked by Blume (1992) for the incompressible case. In what follows, from Eq. (28) we shall derive an explicit expression of the form (30).

We shall apply the eigenprojection method suggested in Xiao et al. (1998). Let $\tau_1, \ldots, \tau_m$ be all the m *distinct* eigenvalues of $\boldsymbol{\tau}$ and $\boldsymbol{P}_1, \ldots, \boldsymbol{P}_m$ the corresponding subordinate eigenprojections of $\boldsymbol{\tau}$. Here, $1 \leqslant m \leqslant 3$. Then we may recast Eq. (28) as

$$\boldsymbol{B} = \sum_{\theta=1}^{m} e^{2(\partial\psi/\partial\tau_\theta)}\boldsymbol{P}_\theta. \tag{31}$$

In the above expression, the eigenprojection $\boldsymbol{P}_\theta$ of $\boldsymbol{\tau}$ subordinate to the eigenvalue τ_θ is uniquely determined by [8] the Sylvester's formula (see, e.g., Xiao et al., 1998, Eq. (1.32))

$$\boldsymbol{P}_\theta = \delta_{1m}\boldsymbol{I} + \prod_{s=1, s\neq\theta}^{m}\frac{\boldsymbol{\tau} - \tau_s\boldsymbol{I}}{\tau_\theta - \tau_s}. \tag{32}$$

When $m = 1$, the continued product in the above is assumed to be zero.

Substituting Eq. (32) into Eq. (31), expressions of the power form (30) may be available for the three cases $m = 1, 2, 3$. The results are as follows.

For $m = 3$, we have Eq. (30) with

$$b_0 = -\frac{1}{\Delta}\left(\tau_2\tau_3(\tau_2-\tau_3)e^{2(\partial\tilde{\Sigma}/\partial\tau_1)} + \tau_3\tau_1(\tau_3-\tau_1)e^{2(\partial\tilde{\Sigma}/\partial\tau_2)} + \tau_1\tau_2(\tau_1-\tau_2)e^{2(\partial\tilde{\Sigma}/\partial\tau_3)}\right), \tag{33}$$

$$b_1 = \frac{1}{\Delta}\left((\tau_2^2-\tau_3^2)e^{2(\partial\tilde{\Sigma}/\partial\tau_1)} + (\tau_3^2-\tau_1^2)e^{2(\partial\tilde{\Sigma}/\partial\tau_2)} + (\tau_1^2-\tau_2^2)e^{2(\partial\tilde{\Sigma}/\partial\tau_3)}\right), \tag{34}$$

$$b_2 = -\frac{1}{\Delta}\left((\tau_2-\tau_3)e^{2(\partial\tilde{\Sigma}/\partial\tau_1)} + (\tau_3-\tau_1)e^{2(\partial\tilde{\Sigma}/\partial\tau_2)} + (\tau_1-\tau_2)e^{2(\partial\tilde{\Sigma}/\partial\tau_3)}\right), \tag{35}$$

where $\widetilde{\Sigma} = \psi(\tau_1, \tau_2, \tau_3)$ and

$$\Delta = (\tau_1-\tau_2)(\tau_2-\tau_3)(\tau_3-\tau_1). \tag{36}$$

[7] Note that $\boldsymbol{\tau}$ and $\boldsymbol{V}$ and hence $\boldsymbol{B}$ are coaxial.

[8] To the contrary, the eigenvectors pertaining to an eigenvalue may be non-unique.

For $m = 2$, we have

$$\boldsymbol{B} = \frac{\tau_1 e^{2(\partial\tilde{\Sigma}/\partial\tau_2)} - \tau_2 e^{2(\partial\tilde{\Sigma}/\partial\tau_1)}}{\tau_1 - \tau_2}\boldsymbol{I} + \frac{e^{2(\partial\tilde{\Sigma}/\partial\tau_1)} - e^{2(\partial\tilde{\Sigma}/\partial\tau_2)}}{\tau_1 - \tau_2}\boldsymbol{\tau}, \tag{37}$$

where $\widetilde{\Sigma} = \psi(\tau_1, \tau_2)$.

For $m = 1$, we have

$$\boldsymbol{B} = e^{2(\partial\tilde{\Sigma}/\partial\tau)}\boldsymbol{I}, \tag{38}$$

where $\widetilde{\Sigma} = \psi(\tau)$ and $\boldsymbol{\tau} = \tau\boldsymbol{I}$.

The three eigenvalues τ_i (possibly repeated) of $\boldsymbol{\tau}$ may be calculated by means of the explicit formula

$$\tau_i = \frac{1}{3}\left(I_1 + \sqrt{6I_2 - 2I_1^2}\cos\frac{\phi - 2i\pi}{3}\right), \quad i = 1, 2, 3, \tag{39}$$

with

$$\phi = \text{arc cos}\left(\frac{8I_1^3 - 36I_1I_2 + 36I_3}{(6I_2 - 2I_1^2)^{3/2}}\right), \tag{40}$$

where the three $I_k = \operatorname{tr}\boldsymbol{\tau}^k$ are the three basic invariants of $\boldsymbol{\tau}$.

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